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Effective Potential of Nambu–Jona-Lasinio Model in Differential Regularization

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ABSTRACT

The method of differential regularization is applied to calculate explicitly the one-loop effective potential of a bosonized Nambu–Jona-Lasinio model consisting of scalar and pseudoscalar fields. The regularization scheme independent relation for the σ mass sum rule is obtained. This method can be readily applied to extended NJL models with gauge fields.

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1. Introduction

A major goal of low-energy hadron physics consists in obtaining effective Lagrangians, which incorporate certain features of quantum chromodynamics that are relevant for describing the properties of hadrons at low and intermediate energies. Among many models, the Nambu–Jona-Lasinio (NJL) model [1] seems to be the simplest pure quark theory, which yields dynamical symmetry breaking and hence a non-vanishing value for the quark condensate. But to make the perturbation series non-divergent, we need to introduce an ultra-violet cut-off as a regularization. Many regularization methods have been employed [2] and it was found that the one-loop effective potentials in various methods can vary substantially with respect to the change of the renormalization scale. In particular, one should know that not all regularization methods are suitable to NJL models. Moreover, higher-loop corrections are also needed to reduce the sensitivity of the renormalization scheme dependence of perturbative results.

In this paper, we shall use a space-time regularization method known as differential regularization (DR) [3] to calculate the one-loop effective potential of a simple NJL model. This method has been shown to be useful to study the quantum corrections in chiral theories. By using DR method, we can determine from the effective potential systematically and unambiguously. It is hope that such a space-time approach can provide a systematic study for extended NJL models [4,5] in which gauge fields are also considered. We

organize the paper as follows. In section 2, we introduce the NJL model and the calculational procedure of DR method. In section 3, we calculate explicitly the one-loop effective action of the NJL model, from which we obtain the one-loop effective potential and determine the σ mass sum rule and the quark condensate.

2. The NJL Model and Differential Regularization

We consider the NJL Lagrangian for scalar and pseudoscalar couplings in the SU(2) sector

$$L_{NJL} = \bar{q}i\gamma^\mu\partial_\mu q + \frac{G}{2} \left[(\bar{q}q)^2 + (\bar{q}i\vec{\tau}\gamma_5 q)^2 \right] - M_0\bar{q}q, \quad (1)$$

where q stands for the isospin SU(2) quark field with 3 colors and current mass $M_0 = m_u = m_d$. G is the coupling constant with mass dimensions $(mass)^{-2}$ in 4 dimensions. $\vec{\tau}$ are the Pauli matrices satisfying $\text{Tr}_\tau[\tau^i\tau^j] = 2\delta^{ij}$. Because of the four-fermion coupling, this Lagrangian is not renormalizable, and for calculating quantum corrections, an ultra-violet momentum cut-off must be introduced.

The quantized theory can be written in terms of a generating functional, which, in the absence of external sources, reads

$$Z_{NJL} = \int D\bar{q}Dq \exp \left[i \int d^4x L_{NJL}(x) \right]. \quad (2)$$

In order to use the model to describe the low-energy properties with the manifest low-energy modes, we bosonize the model with scalar and pseudoscalar

fields σ and $\vec{\pi}$. To do this, we multiply (2) by following gaussian functional

$$1 = \int D\sigma d\vec{\pi} \exp \left[-i \int d^4x \frac{\mu^2}{2} \left(\left(\sigma + \frac{g}{\mu^2} (\bar{q}q - \frac{M_0}{G}) \right)^2 + \left(\vec{\pi} + \frac{g}{\mu^2} \bar{q} i \vec{\tau} \gamma_5 q \right)^2 \right) \right] , \quad (3)$$

which transforms the four-fermion interaction into a Yukawa-like coupling; we obtain

$$\mathcal{Z}_{NJL} = \int D\bar{q} Dq D\sigma D\vec{\pi} \exp \left[i \int d^4x \mathcal{L}_{NJL}(x) \right] , \quad (4)$$

where

$$\mathcal{L}_{NJL} = \bar{q} [i\gamma^\mu \partial_\mu - g(\sigma + i\vec{\pi} \cdot \vec{\tau} \gamma_5)] q - \frac{\mu^2}{2} (\sigma^2 + \vec{\pi}^2) + \frac{M_0 \mu^2}{g} \sigma , \quad (5)$$

with $G = g^2/\mu^2$. In low-energy approximation, we couple the σ field and the pion fields to external sources. The quark fields are not coupled to any external sources and can be integrated over to give a functional determinant, whose evaluation requires a regularization. The calculation can be carried out by using the Feynman rules for the quark fields, such that the one-loop quantum effects are considered by treating quarks as internal lines in the Feynman diagrams.

In this method, it is necessary to determine its vacuum by calculating the effective potential to at least one-loop order. We shall employ differential regularization and carry out the calculation in Euclidean space, so that the functional integral reads

$$Z'_{NJL} = \int D\sigma D\vec{\pi} \exp [-\mathcal{S}_{eff}(\sigma, \vec{\pi})] , \quad (6)$$

where the effective action in the one-loop fermion approximation is

$$\begin{aligned} \mathcal{S}_{eff} = & -\ln \text{Det}[-i\gamma^\mu \partial_\mu + g(\sigma + i\vec{\pi} \cdot \vec{\tau} \gamma_5)] \\ & + \int d^4x \left[\frac{\mu^2}{2}(\sigma^2 + \vec{\pi}^2) - f_\pi m_\pi^2 \sigma \right]. \end{aligned} \quad (7)$$

Note that we have used the PCAC condition $M_0 \mu^2 / g = f_\pi m_\pi^2$, where f_π is the pion decay constant (93 MeV) and m_π is the pion mass (139 MeV).

The functional determinant can be calculated using the Feynman rules, which are given as follows. The massless quark propagator is given by

$$\begin{aligned} \langle q_a^i(x) \bar{q}_b^j(0) \rangle & \equiv S_{ab}^{ij}(x) \\ & = -\frac{i\delta^{ij}\delta_{ab}}{2\pi^2} \frac{\gamma_\mu x^\mu}{x^4}, \end{aligned} \quad (8)$$

with i, j and a, b being isospin and color indices, respectively. The Feynman rule for the $\bar{q}q\sigma$ vertex is $-g\delta^4(x_2 - x_1)\delta^4(x_3 - x_1)$, and that for the $\bar{q}q\vec{\pi}$ vertex is $-ig\vec{\tau}\gamma_5\delta^4(x_2 - x_1)\delta^4(x_3 - x_1)$.

In perturbative calculations, we encounter highly-singular terms of the form

$$\frac{1}{(x^2)^n} \ln^m(\mu^2 x^2), \quad n \geq 2, m \geq 0, \quad (9)$$

where μ is a mass parameter in the problem. The essential idea of differential regularization method is to define these highly-singular terms by

$$\frac{1}{(x^2)^n} \ln^m(\mu^2 x^2) = \underbrace{\square \square \dots \square}_{n-1} G(x^2), \quad x^2 \neq 0, \quad (10)$$

where $G(x^2)$ is a to-be-determined function that has a well-defined Fourier transform and can depend on $2(n-1)$ integration constants, which are the

short-distance cut-offs. In this paper, we encounter only the following two forms:

$$\frac{1}{(x^2)^2} \equiv -\frac{1}{4} \square \frac{\ln x^2 M^2}{x^2}, x^2 \neq 0, \quad (11)$$

$$\frac{1}{(x^2)^3} \equiv -\frac{1}{32} \square \square \frac{\ln x^2 M^2}{x^2}, x^2 \neq 0. \quad (12)$$

where the mass parameter M is an integration constant. Note that we have omitted other irrelevant integration constants for $x^2 \neq 0$.

This regularization method has been used in ϕ^4 theory and QCD and was able to reproduce the well-known results obtained by other methods. The advantage of DR method is that loop corrections in a chiral theory can be calculated unambiguously. We note also that different methods can lead to different results for the one-loop effective potential, which depends strongly on renormalization scheme. Higher-loop corrections can reduce the sensitivity of scheme dependence, but of course, use of a regularization method requires special attention.

3. The One-Loop Effective Action

To obtain the one-loop effective action of the NJL model, we need to evaluate the one-loop self-energies of the σ field and the pion fields, and the vertices of the fields. The self-energy of the σ field with an internal quark loop is easily calculated and reads

$$\Pi_\sigma(x, y) = -g^2 \text{Tr} [S(x - y)S(y - x)]$$

$$\begin{aligned}
&= -\frac{6g^2}{\pi^4} \frac{1}{(x-y)^6} \\
&= \frac{3g^2}{16\pi^4} \square \square \frac{\ln(x-y)^2 M^2}{(x-y)^2}, (x-y)^2 \neq 0, \tag{13}
\end{aligned}$$

where $\square = \square_{(x-y)}$ and we have used $\text{Tr}_\gamma I = 4$, $\text{Tr}_c I = 3$, and $\text{Tr}_\tau I = 2$ for the spinor, color, and isospin degrees of freedom, respectively. The self-energies of the pion fields π^i are

$$\begin{aligned}
\Pi_\pi^{ij}(x, y) &= -g^2 \text{Tr} [i\tau^i \gamma_5 S(x-y) i\tau^j \gamma_5 S(y-x)] \\
&= -\frac{6g^2}{\pi^4} \frac{\delta^{ij}}{(x-y)^6} \\
&= \frac{3g^2}{16\pi^4} \delta^{ij} \square \square \frac{\ln(x-y)^2 M^2}{(x-y)^2}, (x-y)^2 \neq 0. \tag{14}
\end{aligned}$$

One can show that as a consequence of the trace properties of the Dirac gamma matrices in 4 dimensions, the one-loop diagram with one external σ field and one external pion field vanishes. Other non-vanishing one-loop 4-point vertex functions with external boson fields are the following three diagrams: one has 4 σ fields, one has 4 pion fields, and another has 2 σ fields and 2 pions. The one-loop 4-point vertex function with 4 external σ fields is given by

$$\Gamma_{\sigma\sigma\sigma\sigma}(x_1, x_2, x_3, x_4) = G(x_1, x_2, x_3, x_4) + \text{permutations}, \tag{15}$$

where $\Delta_{ij} = x_i - x_j$, and

$$G(x_1, x_2, x_3, x_4) = -g^4 \text{Tr} [S(\Delta_{12}) S(\Delta_{23}) S(\Delta_{34}) S(\Delta_{41})]$$

$$\begin{aligned}
&= \frac{6g^4}{(2\pi^2)^4} \text{Tr}_\gamma [\gamma_\mu \gamma_\nu \gamma_\alpha \gamma_\beta] V_{\mu\nu\alpha\beta} \\
&= \frac{3g^4}{2\pi^8} (V_{\mu\mu\nu\nu} - V_{\mu\nu\mu\nu} + V_{\mu\nu\nu\mu}). \tag{16}
\end{aligned}$$

In terms of $x = x_1 - x_2, y = x_2 - x_3, z = x_3 - x_4$, the function $V_{\mu\nu\alpha\beta}$ reads

$$\begin{aligned}
V_{\mu\nu\alpha\beta} &= \frac{x_\mu y_\nu z_\alpha (x + y + z)_\beta}{x^4 y^4 z^4 (x + y + z)^4} \\
&= \frac{1}{16} \frac{\partial}{\partial x_\mu} \left(\frac{1}{x^2} \right) \frac{\partial}{\partial y_\nu} \left(\frac{1}{y^2} \right) \frac{\partial}{\partial z_\alpha} \left(\frac{1}{z^2} \right) \frac{\partial}{\partial x_\beta} \left(\frac{1}{(x + y + z)^2} \right). \tag{17}
\end{aligned}$$

We are interested in the singular parts of the V functions, which are given by

$$\begin{aligned}
V_{\mu\mu\nu\nu} &\stackrel{sing.}{=} \frac{1}{4} \pi^2 \delta^4(z) \left[\frac{\partial}{\partial x} \left(\frac{1}{x^2} \right) \cdot \frac{\partial}{\partial y} \left(\frac{1}{y^2} \right) \right] \frac{1}{(x + y)^2} \\
&\stackrel{sing.}{=} -\frac{1}{4} \pi^2 \delta^4(z) 4\pi^2 \delta^4(x + y) \frac{1}{x^4} \\
&= \frac{1}{4} \pi^4 \delta^4(z) \delta^4(x + y) \square \frac{\ln x^2 M^2}{x^2}, x^2 \neq 0, \tag{18}
\end{aligned}$$

$$V_{\mu\nu\mu\nu} \stackrel{sing.}{=} \frac{1}{4} \pi^4 \delta^4(y) \delta^4(x + z) \square \frac{\ln z^2 M^2}{z^2}, z^2 \neq 0, \tag{19}$$

$$V_{\mu\nu\nu\mu} \stackrel{sing.}{=} \frac{1}{4} \pi^4 \delta^4(x) \delta^4(y + z) \square \frac{\ln y^2 M^2}{y^2}, y^2 \neq 0. \tag{20}$$

Hence

$$\begin{aligned}
G(x_1, x_2, x_3, x_4) &= \frac{3g^4}{8\pi^4} \left[\delta^4(\Delta_{34}) \delta^4(\Delta_{13}) \square \frac{\ln \Delta_{12}^2 M^2}{\Delta_{12}^2} \right. \\
&\quad - \delta^4(\Delta_{23}) \delta^4(\Delta_{14}) \square \frac{\ln \Delta_{34}^2 M^2}{\Delta_{34}^2} \\
&\quad \left. + \delta^4(\Delta_{12}) \delta^4(\Delta_{24}) \square \frac{\ln \Delta_{23}^2 M^2}{\Delta_{23}^2} \right]. \tag{21}
\end{aligned}$$

The 4-point vertex function with 4 external pion fields can be calculated in the same manner and is given by

$$\Gamma_{\pi\pi\pi\pi}^{(ijkl)}(x_1, x_2, x_3, x_4) = G^{ijkl}(x_1, x_2, x_3, x_4) + \text{permutations}, \quad (22)$$

where

$$\begin{aligned} G^{ijkl}(x_1, x_2, x_3, x_4) &= -g^4 \text{Tr} \left[i\tau^i \gamma_5 S(\Delta_{12}) i\tau^j \gamma_5 S(\Delta_{23}) \times \right. \\ &\quad \left. i\tau^k \gamma_5 S(\Delta_{34}) i\tau^l \gamma_5 S(\Delta_{41}) \right] \\ &= \frac{3g^4}{4\pi^8} (V_{\mu\mu\nu\nu} - V_{\mu\nu\mu\nu} + V_{\mu\nu\nu\mu}) \text{Tr}_\tau [\tau^i \tau^j \tau^k \tau^l] \\ &= \frac{1}{2} G(x_1, x_2, x_3, x_4) \text{Tr}_\tau [\tau^i \tau^j \tau^k \tau^l], \end{aligned} \quad (23)$$

where $\text{Tr}_\tau [\tau^i \tau^j \tau^k \tau^l] = 2\delta^{ij}\delta^{kl} - 2\delta^{ik}\delta^{jl} + 2\delta^{il}\delta^{jk}$. Finally, we list the 4-point vertex functions with mixed external fields:

$$\Gamma_{\pi\pi\sigma\sigma}^{ij\cdot\cdot}(x_1, x_2, x_3, x_4) = G(x_1, x_2, x_3, x_4) \delta^{ij}, \quad (24)$$

$$\Gamma_{\pi\sigma\pi\sigma}^{i\cdot j\cdot}(x_1, x_2, x_3, x_4) = -G(x_1, x_2, x_3, x_4) \delta^{ij}, \quad (25)$$

$$\Gamma_{\pi\sigma\sigma\pi}^{i\cdot\cdot j}(x_1, x_2, x_3, x_4) = G(x_1, x_2, x_3, x_4) \delta^{ij}, \quad (26)$$

$$\Gamma_{\sigma\pi\pi\sigma}^{\cdot i j\cdot}(x_1, x_2, x_3, x_4) = G(x_1, x_2, x_3, x_4) \delta^{ij}, \quad (27)$$

$$\Gamma_{\sigma\pi\sigma\pi}^{\cdot i\cdot j}(x_1, x_2, x_3, x_4) = -G(x_1, x_2, x_3, x_4) \delta^{ij}, \quad (28)$$

$$\Gamma_{\sigma\sigma\pi\pi}^{\cdot\cdot ij}(x_1, x_2, x_3, x_4) = G(x_1, x_2, x_3, x_4) \delta^{ij}, \quad (29)$$

where $G(x_1, x_2, x_3, x_4)$ is given by (21).

Higher-point vertex functions either vanish or do not provide logarithmic contributions and are omitted here.

Summing all the logarithmic contributions, we get (up to quartic in the fields) the one-loop effective action before renormalization

$$\begin{aligned}
\Gamma^{(1)}[\sigma, \vec{\pi}] = & \frac{-3g^2}{32\pi^4} \int d^4x d^4y (\Box \Box \frac{\ln(x-y)^2 M^2}{(x-y)^2}) [\sigma(x)\sigma(y) + \vec{\pi}(x) \cdot \vec{\pi}(y)] \\
& - \frac{3g^4}{32\pi^4} \int d^4x d^4y (\Box \frac{\ln(x-y)^2 M^2}{(x-y)^2}) \left[2\sigma^3(x)\sigma(y) - \sigma^2(x)\sigma^2(y) \right. \\
& \quad + 2\vec{\pi}^2(x)\vec{\pi}(x) \cdot \vec{\pi}(y) - \vec{\pi}^2(x)\vec{\pi}^2(y) \\
& \quad - 2\vec{\pi}^2(x)\sigma^2(y) + 2\vec{\pi}^2(x)\sigma(x)\sigma(y) \\
& \quad \left. + 2\vec{\pi}(x) \cdot \vec{\pi}(y)\sigma^2(x) \right]. \quad (30)
\end{aligned}$$

Now using the fact that $\Box \frac{1}{x^2} = -4\pi^2 \delta^4(x)$, we get

$$\begin{aligned}
\frac{\partial \Gamma^{(1)}[\sigma, \vec{\pi}]}{\partial t} = & \frac{3g^2}{8\pi^2} \int d^4x \left[(\partial_\mu \sigma(x))^2 + (\partial_\mu \vec{\pi}(x))^2 \right] \\
& + \frac{3g^4}{8\pi^2} \int d^4x \left(\sigma^2(x) + \vec{\pi}^2(x) \right)^2, \quad (31)
\end{aligned}$$

where $t = \ln(M^2/\Lambda^2)$ with Λ being the renormalization scale. From the general form of the effective action in Euclidean space

$$\Gamma(\sigma, \vec{\pi}) = \int d^4x \left[+V_{eff}(x) + \frac{Z}{2} \left((\partial_\mu \sigma(x))^2 + (\partial_\mu \vec{\pi}(x))^2 \right) + \dots \right], \quad (32)$$

where Z is a renormalization constants, we obtain the effective potential

$$\begin{aligned}
V_{eff}(x) = & \frac{\mu^2}{2} (\sigma^2(x) + \vec{\pi}^2(x)) - f_\pi m_\pi^2 \sigma \\
& + \frac{3g^4}{8\pi^2} \left(\sigma^2(x) + \vec{\pi}^2(x) \right)^2 \ln \left(\frac{M^2}{\Lambda^2} \right). \quad (33)
\end{aligned}$$

The NJL model has a nontrivial vacuum, which has non-vanishing vacuum expectation values (VEV's) for the σ and $\vec{\pi}$ fields. These VEV's are determined by the stationary phase conditions :

$$\left. \frac{\partial V_{eff}}{\partial \sigma} \right|_{\text{vacuum}} = \left. \frac{\partial V_{eff}}{\partial \vec{\pi}} \right|_{\text{vacuum}} = 0, \quad (34)$$

from which we get $\pi_v^i = \langle \pi^i \rangle = 0$ and $\sigma_v = \langle \sigma \rangle$ can be determined from the equation

$$\mu^2 \sigma_v - f_\pi m_\pi^2 + \frac{3g^4}{2\pi^2} \sigma_v^3 \ln \left(\frac{M^2}{\Lambda^2} \right) = 0. \quad (35)$$

Alternatively, we can obtain σ_v from the mass equations for the fields. The masses are related to the curvatures of the effective potential at the vacuum point; for the pion fields,

$$m_\pi^2 \equiv \left. \frac{\partial^2 V_{eff}}{\partial \pi^{i2}} \right|_{\text{vacuum}} = \mu^2 + \frac{3g^4}{2\pi^2} \sigma_v^2 \ln \left(\frac{M^2}{\Lambda^2} \right), \quad (36)$$

while for the σ field,

$$m_\sigma^2 \equiv \left. \frac{\partial^2 V_{eff}}{\partial \sigma^2} \right|_{\text{vacuum}} = \mu^2 + \frac{9g^4}{2\pi^2} \sigma_v^2 \ln \left(\frac{M^2}{\Lambda^2} \right), \quad (37)$$

where we have $\sigma_v = f_\pi$.

If we demand that the first non-vanishing term yield the proper kinetic energy of the mesons, we set $Z = \frac{3g^2}{4\pi^2} \ln(M^2/\Lambda^2) = 1$, which reduces the number of arbitrary parameters, we obtain the regularization scheme independent relation for the σ -mass sum rule [2],

$$m_\sigma^2 = 4g^2 f_\pi^2 + m_\pi^2. \quad (38)$$

In addition, from the equation of motion for the σ field with $M_0 = 0$, and the VEV $\sigma_v = f_\pi$, we obtain in the soft pion limit, a cut-off independent relation,

$$M_0 < \bar{q}q > = -f_\pi^2 m_\pi^2, \quad (39)$$

where the value of the quark condensate $< \bar{q}q >$ provides a measure of spontaneous symmetry breaking.

4. Discussion

In this paper, we have calculated explicitly the effective potential at the one-loop order in DR, and thus provided a systematic spacetime approach, which can be readily applied to more useful extended NJL models [4,5] in which gauge fields also play a role.

We should note that in ref. [2], several regularization schemes were used to obtain one-loop effective potentials, which were shown to differ substantially with respect to renormalization scales. This shows that the regularization methods are not all directly applicable to chiral models like the NJL models, and the use of those methods should be used with care. It should be mentioned that higher-loop corrections are generally needed to reduce the sensitivity of renormalization scheme dependence of physical quantities. Therefore, it is of practical importance also to provide a calculational procedure for two- or higher-loop orders.

Finally, we should mention that in our calculation we have used the same short-distance cut-off parameter M for regularizing the divergences. This ambiguity can be avoided if we employ the background-field method with the symmetry of the model being enforced throughout the calculation, such that only one cut-off parameter may be allowed.

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